THE INTERPOLATION PROBLEM

Most early programs (and far too many recent ones) have taken a purely metric approach to the interpolation (or contouring) problem for arbitrarily distributed data points. They collected all data points within a given distance from the estimation point, or the nearest six, or other variants intended to select both a "reasonable" number of data points, and also to select a "surrounding" set. Most of these approaches could be shown to be inadequate by the judicious selection of awkwardly distributed data sets. A subsequent approach involved the triangulation of the data set, followed by a local polynomial (or linear) patch within each triangle. Two problems arose with this: firstly many triangulations of a given data set were possible, and the same set could not be guaranteed if the data were re-entered in a different order; and secondly local patches, which were defined to merge with adjacent triangles with a given level of continuity (usually slope) were therefore discontinuous at higher levels (usually curvature) - and this was very visible in the resulting map.

The first of these problems was resolved some time ago with the general agreement that the Delaunay triangulation was the appropriate method - it was globally stable, independent of input order, and a local perturbation (moving a point, inserting a new one or deleting an old one) produced only a local modification of the network. This followed from the definition, as the Delaunay triangulation was the dual of the Voronoi diagram, as in Figure 1, where each point of a data set from Davis (1973) produced a "bubble" or "zone of influence" representing all parts of the map closer to that data point than to any other. Thus modifying one point or zone would affect the boundaries of only the immediately-adjacent Voronoi polygons, and points were considered adjacent only if their Voronoi polygons touched. One well-known algorithm for generating the point Voronoi diagram was described by Green and Sibson (1978).

The second problem, that of fitting an appropriate polynomial patch, was handled by reverting to an older, simpler approach: the weighted average method - but how to select the weights? By inserting the estimation point into the data-structure (the Delaunay triangulation or equivalent Voronoi diagram) its neighbours were defined in the same way as for the data points in
Figure 1. Thus any neighbouring Voronoi polygons were by definition neighbours to the Voronoi polygon of the estimation point - hence the neighbouring points were defined. The weights to be used were as easily defined - if the Voronoi polygon of a data point was adjacent to (had a common boundary with) the Voronoi polygon of the estimation point, then the insertion of that estimation point must have stolen some of the original area of the data point's polygon. This stolen area formed the basic weighting assigned to each neighbouring data point. Figure 2 illustrates the method. Part a) shows a simple data-set with Voronoi polygons and the dual triangulation. "" marks the spot where an estimation point is to be inserted. Part b) shows the new Voronoi polygon formed, and part c) shows the proportions of the adjacent areas stolen by the new polygon. Of course, once the areas are calculated the sampling point is deleted from the data set, as its function is purely temporary. It should be noted here that Sibson (1982) developed a polynomial interpolation procedure based on Voronoi sub-tiles that he called "natural neighbour interpolation". This writer's work using weighted average methods was developed independently in a commercial context (see Gold, 1989, for more details).

THE ADVANTAGES OF THEFT

This "area-stealing" or "theft" spatial model eliminates many problems in conventional weighted-average techniques that are due to the poor specification and negative interaction of separately-defined neighbour selection and weighting processes. Used in unmodified form it produces the two-dimensional equivalent of linear interpolation between a set of points in one spatial dimension. Thought of as an operation on polygons, where each original data point's elevation is assigned to the whole polygon, the area-stealing model is a volume-preserving process. So far undefined by this approach are "smooth" interpolation, where the surface varies smoothly over the data points themselves as well as over the rest of the map, and any measure of the error of the estimate. The first issue is described further in Gold (1989).

STATISTICS - VARIANCE OF THE ESTIMATE

When we delete or insert the sampling point, and use the stolen area technique, we obtain a weighted average of the individual estimates of the neighbours. From this it should be possible to make some statistical statements about the reliability of the stolen-area estimate. This may readily be done as a simple weighted variance calculation - but only by ignoring the autocorrelation between spatial observations. Herein lies the major problem in spatial statistics.

The problem is facilitated in our particular case by our prior assumption of Voronoi adjacency: all our observations are associated with contiguous polygons. The spatial autocorrelation associated with polygon sets (or the dual networks) has been well studied in geography -- e.g. Cliff and Ord (1973,1981) -- for a
network with known weightings. Our polygon adjacency gives the network; our area-stealing gives the weighting, by deleting and reinserting the data points one at a time and saving the "area-stealing" weights. From this a weighting matrix can be constructed, giving the weighting of every point against every other.

Depending on the time-series model assumed, an intermediate variance matrix (BBT) can be generated from the local (or global) weighting matrix or network (see Table 1). This gives a clustering/sampling variance estimate. When a Maximum Likelihood variance estimate for the attribute data (e.g. elevation) is included, a variance/covariance matrix (Table 1) can be generated about each data point. This therefore includes both the "elevation" variability as well as the "spatial clustering" variability, and from this can be generated a probability envelope (e.g. 95%) for the elevation at each data point.

In order to produce a contoured "reliability map" the envelope values at each data point may be contoured conventionally. By doing this we are, unfortunately, again assuming zero autocorrelation of observations around the sampling point used for area-stealing interpolation. This may be overcome by using the method of linear composites (Table 1), whereby we may take a weighted average of the variance/covariance matrices at the neighbouring data points. By using the same area-stealing weightings for the matrices as we do for the interpolation itself, this procedure becomes almost "common sense". Figure 3 shows the Davis data of Figure 1 contoured, and Figure 4 shows the standard deviation obtained by the linear composite calculation.

There are, indeed, some tricky bits in all this, concerning the choice of the time-series model and the localization of the variance/covariance matrix. At present it seems that if we want to create our matrix using only the central data point and its immediate neighbours, this is only achievable using the moving average model, as the variance/covariance matrix generated by the autoregressive model shows strong correlations between data points that are close neighbours neither to each other or to the central data point of interest. Luckily the moving average matrix has correlations that decay to zero very rapidly when moving away from the central point.

The result of all this is an error envelope (and variance/covariance matrix) definable at any point in the map space. Depending on the localization process used to extract a portion of the overall matrix, the values are continuous or "almost" continuous over the whole map, based only on the observations at the data points. Since the area-stealing weighted average interpolation process may be used on the Voronoi polygons of any map object, not just points, the recent implementation of Voronoi diagram construction for both points and line segments (Gold, in press), and the ability to interpolate surfaces based on these data objects, suggests some interesting extensions to the usual idea of surface error estimation. Figure 5 shows the Voronoi
diagram for a single line segment in a point set, and Figure 6 shows the Voronoi regions for a polygon constructed from points plus boundary line segments, including an interior island. The area-stealing interpolation methodology is appropriate for any problem where the Voronoi (or equivalent) polygons may be generated in the presence and absence of the sampling point.

VARIANCE OF MISSING CENSUS DISTRICT ESTIMATES

For the moment, one example of the extended interpolation problem will suffice: the missing census-tract problem. Suppose we have a census study of a set of contiguous administrative districts. One of these is found to have invalid or missing data -- how do we estimate the missing value?

One consequence of the ability to generate Voronoi regions for points plus line segments is that, for any map polygon constructed from these inserted points and line segments, the "skeleton" of the polygon is defined by the Voronoi region boundaries. These boundaries represent the Voronoi extensions of the neighbouring polygons into the polygon with the missing data -- just as if the polygon itself, and not just its data, were missing. Given our previous area stealing interpolation method, we may clearly interpolate the value of the missing census district by "stealing back" these areas taken over by the neighbouring polygons. This gives us a weighted average estimate, as before. Figure 7 shows the technique applied to a problem examined by Tobler and Kennedy (1986), for the State of Kansas.

We can take this one step further. In order to obtain our error estimates based on interpolation of a point data set we had to delete and re-insert each data point once -- which gave us our weighting functions between neighbours, from which we derived our interpolated value and error estimate. In the case of the missing census district problem the same process may be applied, giving a set of weightings between adjacent polygons, and hence an interpolated value. Using again the method of linear composites, an error estimate may be obtained for the interpolated value, thus showing the generality of the Voronoi statistical techniques just described.

THE POINT IN POLYGON PROBLEM

We will close with a "trivial" application. As previously mentioned, the (interior and exterior) skeleton of a map polygon is formed by the boundaries of the Voronoi diagram of the map polygon's boundary points and line segments. Let us suppose that after the construction of the map polygon and its Voronoi diagram a labelling procedure has been executed that labels (i.e. gives a value to) all Voronoi regions that form the interior of the polygon. (The Voronoi region of a line segment consists of two parts, one associated with each side of the line segment.) If a sampling point is located within the polygon and the area-stealing process applied, all the non-zero weights will be associated with
the **same** polygon label (value). This is illustrated by point '? inside the island in Figure 6. Thus the "interpolated" result of the point-in-polygon query will be 100% of the desired polygon label! Similarly, any variance estimate based on the area-stealing method will equal zero, as all "elevations" are the same. While these results are trivial, they do indicate that the point in polygon problem is an interpolation problem, and also that if the polygon value represented an "elevation" (or census result), the same interpolation and statistical methods derived previously are applicable to such "stepped" surfaces.

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