Further Research on Voronoi Diagrams

- a Common Basie for Many Applications -

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INTRODUCTION

Voronoi diagrams are currently being studied extensively by those in the field of computational geometry, primarily in terms of the order of efficiency of algorithms. Other criteria, however, may be of as much significance for spatial and data-base applications. This paper will discuss some of the basic operations on points and line segments as well as some of the G.I.S. operations that may be performed with three techniques.

A “general proposition” is first stated, giving the relation of
Voronoi diagrams to both geometry and graph theory, and the implications of this. The second part of the paper consists of a series of statements about spatial data structures, starting with the writer's early work and building on that. Many of the diagrams illustrating the various points are available from these references.

GENERAL PROPOSITION

By a "mapping" problem is meant here any desired process that may be applied directly to a map or maps, inclusive of geometric or topologic processes but exclusive of any particular data-base operations on attributes. By introducing computers, with their discrete logic to these particular problems we inevitably introduce various strengths and weaknesses into our basic operating environment. Because of the discrete nature of the machine, discrete mathematics, especially graph theory, becomes an increasingly valuable tool. Recent work in computational geometry, although of an increasingly theoretical nature, provides many of the basic insights into the feasibility of various algorithms for geometric operations. Of particular interest are studies of the generation of various classes of
Voronoi diagrams. One particular weakness of discrete computers is well-known: the inability to provide exact results for line intersection and similar problems. Consequently geometric (co-ordinate) operations can not guarantee consistent graphical structures (topology) except with considerable care.

It is proposed here that use of Voronoi diagrams, especially Euclidean-distance nearest-object Voronoi diagrams of points and line-segments in the plane, permits a general-purpose conversion of geometric information to a graphically-structured form amenable thereafter to graph-traversal and other fundamental discrete operations appropriate to the computing environment employed.

Two additional points may be noted. Voronoi diagrams may be created by the “divide-and-conquer” techniques beloved of computational geometers. These provide the ultimate in theoretical efficiency but are appropriate only for the initial building of large data sets. While theoretically sound, they are often extremely complex to program - and in many cases they have not been implemented at all. Object-at-a-time methods are theoretically less efficient but are easier to
implement and lend themselves more readily to processes that require the modification of only a small part of the map. Their efficiency loss is entirely in the search-for-the-nearest-existing-object step, which can readily be improved from the worst-case scenario by a variety of practical steps. Since the Voronoi diagram construction is intended to convert from a geometric to a graph-based format, it must itself avoid inconsistencies due to the discretization of continuous co-ordinates. Care in the order of arithmetic operations in the basic tests can eliminate this concern.

In the implementation of graph-based structures for map operations it is necessary, for large data-sets, to maintain node-and-pointer data structures internally, where the nodes represent map objects (points or lines) and the pointers represent adjacency relationships between objects. In a one dimensional example, of an ordered set of numbers, this would be implemented as a linked list. Basic operations on this linked list are: initialize; search; insert; delete and possibly for some applications a "swicht" operation between adjacent
nodes. The search operation would be the most inefficient (order $n^{82}$) if left as a linear search; this could be improved to order $n \log(n)$ if a tree structure were built from the nodes using conventional data-structure methods. This is directly comparable to the two-dimensional case, where the nodes and pointers represent a planar graph - in this particular approach a triangulation. The same basic operations are required: initialize; search; insert; delete and switch (of two adjacent triangles - see Gold 1978). As before, a more efficient search operation can be generated based on a tree structure if required. Once the Voronoi generator has constructed this graph data structure, conventional graph traversal algorithms (e.g. depth-first and breadth-first searches, shortest path and minimum-spanning tree construction) may be used to answer a variety of basic geographic queries.

BASIC OPERATIONAL STATEMENTS

The following statements form the conceptual stage in the design of an operational system to handle a variety of
spatial data processing needs.

1. Polygons are formed from interconnected vertices and edges. In order for a polygon to be topologically complete pointers must exist between vertices (if defined) and edges. The resulting region on the graph must then be labelled.

2. All vertices (nodes) in a polygonal map can be forced to have a valence of three by creating an imaginary zero-length edge and splitting the original node.

3. The dual of the modified polygon set is a triangulation, where all polygons are represented by nodes and all vertices have become triangles. The original arbitrary boundaries between adjacent polygons are replaced by triangle edges representing an adjacency relationship between two polygons. (See Gold 1987b.)

4. Triangulations may readily be stored as fixed-
length records storing the three vertices, the three adjacent triangles and, if required, the three bounding edge record numbers for each triangle. (See Gold et al. 1977 or Gold 1978.)

6. An alternative to a triangulation as a basic record type is a line segment. This is also of fixed length, storing pointers to the two end vertices and the two (anticlockwise) adjacent line segments. Both line segments and triangulations are valid data structures whose relative advantages are minor and depend on the application. (See Gold, in press.)

6. Triangulations in this context express relationships between triples of objects - in this case polygons.

7. If triangulations express adjacency relationships between points (the duals of polygone) then the Delaunay triangulation is an appropriate expression of the adjacency relationships between Voronoi polygons and is thus an expression of the adjacency relations between the original generating data points.
8. The objects associated with the triangle vertices need not be points - they may be any objects, in particular in this application points plus line segments. (See Gold 1987b.)

9. The Voronoi criterion for any object is defined in the same manner as for points, and may readily be calculated. Between a point and a line it is a parabola.

10. The boundaries between Voronoi polygones are implicit in the relationship between any two adjacent vertices (objects) in the triangulation, and need not be preserved. The centre of the triangle (i.e. the junction between three Voronoi boundaries) is more critical in determining which boundaries are to be preserved to form the triangulation. In this case a triangulation structure appears preferable to a line-segment data structure. (See Gold 1987b.)

11. Basic operations for linked lists are: initialize; insert; delete; search; and switch. The equivalent for triangulation are: initialize (create a bounding
triangle to enclose the data set; search (walk through the triangulation to find the bounding triangle for the point or object); insert (split the bounding triangle into three to accommodate the new object); switch (interchange the diagonal on adjacent triangle pairs - performed if the Voronoi criterion is not met for the current triangle pair); delete (remove an object from the triangulation by temporarily merging two adjacent objects and deleting the two redundant triangles - the reverse of insert). (See Gold, 1987s.)

12. The switch operation is performed whenever the common boundary between two adjacent triangles does not conform to the Voronoi criterion. Upon any insert or delete operation the triangle edges adjacent to the affected area are stored for subsequent testing. The Voronoi criterion guarantees that only a limited local number of edges need be tested.

13. The insert and “delete” operations are equivalent to “split” and “merge” operations on objects. This permits the hierarchical organizing of objects into a tree
structure if required, for efficient organization or searching. This is most easily understood if the dual of the objects - a polygon set - is considered. In this mode two adjacent polygons A and B are merged into polygon AB. If deletion is the purpose either A or B are removed and polygon AB retains the name of B or A. In a hierarchical binary tree structure the parent node AB has both the leaves A and B deleted and becomes itself a leaf labelled either A or B. Where the reversal is required and a new polygon (node) B is to be created from the original node A, the polygon A is relabelled AB in the binary tree and A and B are preserved as leaves. This is equivalent to splitting polygon A, or to creating a new node B adjacent to node A. In all cases once objects have been inserted or deleted the switch function is used as described above to ensure adherence to the Voronoi criterion. (See Gold 1987b.)

14. AB described in Gold (1987c) interpolation may be performed by the judicious insertion and deletion of dummy sampling points in order to determine the relative areas of the adjacent Voronoi polygons stolen by the new dummy point. From this appropriate interpolants may be

15. As described in (Gold 1987b) line segments are constructed from their two end points and a connecting link. If these end points and the line segments are inserted into the Voronoi network they will each generate their own Voronoi region. For line segments connected to form a polygon, the interior boundaries of these regions form the skeleton or medial axis transform of the polygon in vector space.

16. As described in Gold and Maydell (1978), Gold (1987a) and Gold and Cormack (1986 and 1987) any triangulation may be processed as an ordered binary tree with respect to some viewpoint, permitting front-to-back or radially-outwards ordering of objects on a map. This is of use in contour construction, pen movement minimization, hidden line or surface removal and the searching for all nearest neighbours within some tolerance, as well as many other applications.

In conclusion, the concept of Voronoi diagram construction
for point and line objects provides a good general framework for a wide variety of spatial data processing applications.

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REFERENCES


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